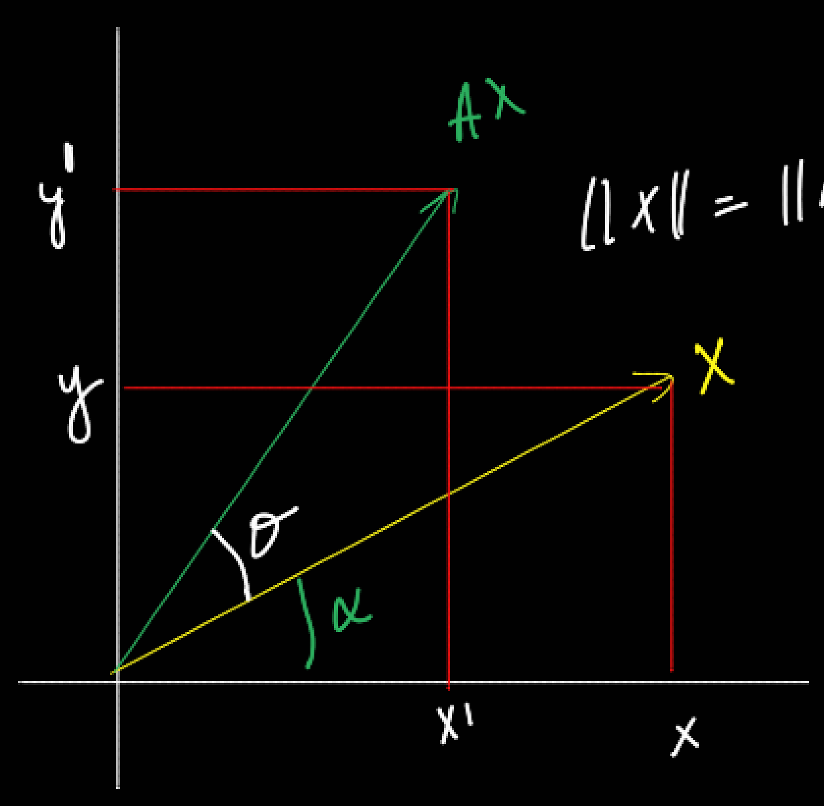
ROTATION MATRIX A



$$\|x\| = \|Ax\| = d$$

$$x = d\cos\alpha \qquad y = d\sin\alpha$$

$$x' = d\cos(\theta+\alpha) \qquad y' = d\sin(\theta+\alpha)$$

$$x' = d[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = x\cos\theta - y\sin\theta$$

$$y' = d[\sin\theta\cos\alpha + \cos\theta\sin\alpha] = x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$A =$ skew symmetric
$|A| = 1$
No real eigen value & eigen vectors

Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \operatorname{adj}(\operatorname{adj} M)$, then which of the following statements is/are ALWAYS TRUE?

(A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\operatorname{adj} M)^2 = I$

$$M^{-1} = \operatorname{adj}(\operatorname{adj} M)$$

$$M^{-1} = \operatorname{adj}[|M|\cdot M^{-1}]$$

$$M^{-1} = |M|^2\cdot\operatorname{adj}(M^{-1})$$

$$M^{-1} = |M|^2\cdot|M^{-1}|\,M$$

$$I = |M|\cdot M^2 \rightarrow (a)$$

$$\det I = |M|^n\cdot|M|^2$$

$1 = |M|^{n+2} \longrightarrow$ only solution is $|M| = 1$

B true

$\operatorname{adj} M = |M|\,M^{-1} \rightarrow$ if M is invertible

where $|M| \rightarrow$ determinant

$\operatorname{adj} K^{-1} = |K|\cdot K \rightarrow$ for invertible K

apply in (a) to get $I = M^2 \rightarrow$ C true

$$\operatorname{adj}(M)^2 = (|M|\cdot M^{-1})^2$$

$$\operatorname{adj}(M)^2 = |M|\cdot(M^2)^{-1}$$

$(\operatorname{adj} M)^2 = 1\cdot I = I \rightarrow$ D true

$$\operatorname{Adj}(kA) = |kA|\,(kA)^{-1}$$

$$= k^n|A|\,\frac{1}{k}A^{-1} = k^{n-1}|A|\cdot\frac{\operatorname{Adj}(A)}{|A|} = k^{n-1}|A|$$

$$\boxed{\therefore \operatorname{Adj} kA = k^{n-1}\operatorname{Adj}(A)}$$

Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that $A^2 = \mathbf{0}$.

Suppose any vector in the column space of A can be expressed as $\vec{v} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$

Which of the following statements is/are necessarily true?

✓ A. If $\vec{v} \in \mathrm{Col}(A)$, then $\vec{v} \in \mathrm{Null}(A)$

✓ B. $\mathrm{Col}(A) \subseteq \mathrm{Null}(A)$

✓ C. $\mathrm{rank}(A) \leq \frac{n}{2}$

D. $\mathrm{Col}(A) = \mathrm{Null}(A)$

$A \cdot A = 0$

$\Rightarrow A \times (\text{Column 1 of } A) = 0$

$A \times (\text{Column 2 of } A) = 0$

$\vdots$

$A \times (\text{Column } n \text{ of } A) = 0$

All columns of A are also in Nullspace of A

Colums span column space

All colums are in nullspace

$\Rightarrow$ if $X \in C(A) \Rightarrow X \in N(A)$

$\dim(C(A)) \leq \dim(N(A))$

$r \leq n - r$

$2r \leq n$

$r \leq n/2$

$r \leq \lfloor n/2 \rfloor$

if $n = $ odd

$\text{Max dim}[C(A)] = r_{max} = \frac{n-1}{2}$

$\text{Min dim}[N(A)] = n - r = n - \frac{(n-1)}{(2)} = \frac{2n-n+1}{2} = \frac{n+1}{2}$

both dimensions don't match $\Rightarrow$ option D False

option D — counterexample

if $X \in N(A) \xrightarrow{?} X \in C(A)$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

rank $A = 1$

$\dim C(A) = 1$

$\dim N(A) = 2$

$N(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$C(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow C(A) \subseteq N(A)$ not equal

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & -7 & 2 \\ -12 & -24 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -3 & 1 \\ -4 & -7 & 2 \\ -16 & -30 & 9 \end{bmatrix}$$

Which of the following statements is/are correct?

A. Matrices A and B have the same characteristic polynomial.
B. A and B have the same rank.
C. A and B are similar matrices.
D. Exactly one of A, B is diagonalizable.

$-(112+60+108)$

$-1 \quad -3$
$-4 \quad -7$
$-16 \quad -30$

$63 + 96 + 120$

$|B| = 8+3-12$
$|B| = -1$

A, B trace $= 1$
$|A| = -1$

Both non zero determinant $\Rightarrow$ Full rank (B)

C.E equal since trace, determinant & $\sum$ Prin. Minor are same $\Rightarrow$ (A)

Principal Minor A
$M_{11} = -1$
$M_{22} = 7$
$M_{33} = -7$
$\sum M_A = -1$

Principal Minor B
$M_{11} = -3$
$M_{22} = 7$
$M_{33} = -5$
$\sum M_B = -1$

C.E $\Rightarrow \lambda^3 - \text{trace}\,\lambda^2 + (\sum M_{ii})\lambda - \det = 0$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$

$(\lambda-1)(\lambda^2-1) = (\lambda-1)(\lambda-1)(\lambda+1) = \underline{+1, +1}, -1$ repeats

$(A - I)X = 0 \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ -4 & -8 & 2 \\ -12 & -24 & 6 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

nullity of $(A-I) = 2$ } 2 distinct eigen vectors

$(B-I)X = 0 \quad \begin{vmatrix} -2 & -3 & 1 \\ -4 & -8 & 2 \\ -16 & -30 & 8 \end{vmatrix} \rightarrow$ nullity $= 1 \Rightarrow$ only 1 distinct eigen vectors

$\rightarrow$ A & B not similar
only A is diagonalisable $\Rightarrow$ D

Fastest way to get orthonormal eigen vectors given repeated eigen values.

Take above matrix A $\rightarrow$ for $\lambda = +1$, AM $= 2$, GM $= 2$

① Solve $(A - \lambda I)X = 0$

$$\begin{vmatrix} 0 & 0 & 0 \\ -4 & -8 & 2 \\ -12 & -24 & 6 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

nullity $2 \Rightarrow$ 2 free Variable

$\Rightarrow x_3 - 4x_2 - 2x_1 = 0$

$x_3 = 0, x_2 = 1 \Rightarrow x_1 = -2$

$x_3 = 2, x_2 = 0 \Rightarrow x_1 = 1$

$\underset{A}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$

② given eigen vectors $\rightarrow$ apply gram schmi

$a \rightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad a = \frac{A}{\|A\|}$

$b' \rightarrow = B - \bar{A}\frac{A^T B}{A^T A}$

$= B - \bar{a}(\bar{a}^T B)$

$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{5}} \cdot \frac{-2}{\sqrt{5}}$

$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}}{5}$

$b' = \begin{bmatrix} 1/5 \\ 2/5 \\ 10/5 \end{bmatrix}$

Nullity of $[A - \lambda I] = $ GM of (λ)

special case Nullity of A $=$ GM of $(\lambda = 0)$

Review of block matrices

A block matrix (also called partitioned matrix) is a matrix of the kind

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are matrices, called blocks, such that:

- A and B have the same number of rows;
- C and D have the same number of rows;
- A and C have the same number of columns;
- B and D have the same number of columns.

Ideally, a block matrix is obtained by cutting a matrix vertically and horizontally. Each of the resulting pieces is a block.

Example The matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 5 & 0 & 7 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

can be written as a block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 2 & 5 \\ 2 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 7 \\ 2 & 2 \end{bmatrix}$$

Example The matrix

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 0 \\ 8 & 2 & 2 \end{bmatrix}$$

can be written as a block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where

$$A = [1] \quad B = \begin{bmatrix} 0 & 3 \end{bmatrix}$$
$$C = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 2 & 2 \end{bmatrix}$$

An important fact about block matrices is that their multiplication can be carried out as if their blocks were scalars, by using the standard rule for matrix multiplication:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$$

The only caveat is that all the blocks involved in a multiplication (e.g., AE, BG, CE) must be conformable. For example, the number of columns of A and the number of rows of E must coincide.

Determinant of a block-diagonal matrix with identity blocks

A first result concerns block matrices of the form

$$\Gamma = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

or

$$\Gamma = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$$

where I denotes an identity matrix, 0 is a matrix whose entries are all zero and A is a square matrix.

Block matrices whose off-diagonal blocks are all equal to zero are called block-diagonal because their structure is similar to that of diagonal matrices.

Not only the two matrices above are block-diagonal, but one of their diagonal blocks is an identity matrix. We will call them **block-diagonal matrices with identity blocks**.

The following proposition holds.

Proposition Let Γ be one of the two block-diagonal matrices with identity blocks defined above. Then,

$$\det(\Gamma) = \det(A)$$

Proof

Determinant of a block-triangular matrix

A block-upper-triangular matrix is a matrix of the form

$$\Gamma = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where A and D are square matrices.

Proposition Let Γ be a block-upper-triangular matrix, as defined above. Then,

$$\det(\Gamma) = \det(A)\det(D)$$

Proof

A block-lower-triangular matrix is a matrix of the form

$$\Gamma = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

where A and D are square matrices.

Proposition Let Γ be a block-lower-triangular matrix, as defined above. Then,

$$\det(\Gamma) = \det(A)\det(D)$$

Proof

The general case

We can now prove the general case, by using the results above.

Proposition Let Γ be a block matrix of the form

$$\Gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are square matrices. If A is invertible, then

$$\det(\Gamma) = \det(A)\det(D - CA^{-1}B)$$

Proof

Proposition Let Γ be as above. If D is invertible, then

$$\det(\Gamma) = \det(A - BD^{-1}C)\det(D)$$

$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ Then $\det(M) = \det(A^2 - B^2)$

$M^T = \begin{bmatrix} A^T & B^T \\ B^T & A^T \end{bmatrix}$

M – symmetric only if A & B are symmetric

$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \Rightarrow \begin{bmatrix} I & I \\ I & -I \end{bmatrix} M \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} A+B & A-B \\ A+B & -(A-B) \end{bmatrix}$ $\begin{bmatrix} 2(A+B) & 0 \\ 0 & 2(A-B) \end{bmatrix}$

$(-2)\cdot\det(M)\cdot(-2) = 4\det(A+B)\det(A-B)$

$\boxed{\det(M) = \det((A+B)(A-B))}$

$\det M = \det(A^2 - B^2 - AB + BA)$

Let R be the row reduced echelon form of a 4×4 real matrix A and let the third column of R be $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$.

Consider the following statements:

- P: If $\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ 0\end{bmatrix}$ is a solution of $Ax = 0$, then $\gamma = 0$.
- Q: For all $b \in \mathbb{R}^4$, $\text{rank}[A \mid b] = \text{rank}[R \mid b]$

Then

A. Both P and Q are TRUE
B. P is TRUE and Q is FALSE
C. P is FALSE and Q is TRUE
D. Both P and Q are FALSE

Your Answer: B | Correct Answer: D | Incorrect | Time taken: 04min 48sec | Discuss

$A = \begin{bmatrix}1&0&0&a\\0&1&1&b\\0&0&0&1\\0&0&0&0\end{bmatrix}$ Counter

$A - \begin{bmatrix}1&1\\1&1\end{bmatrix}$ $R = \begin{bmatrix}1&1\\0&0\end{bmatrix}$

$[A|b] = \begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}$ $[R|b] = \begin{bmatrix}1&1&1\\0&0&1\end{bmatrix}$

Rank = 1 Rank = 2

RREF → Maintains Row space
→ Null space
→ pivot columns
☆ → if a column is not pivot column, then it is linearly dependent on all columns before it, it may or may not be dependent on columns after it.

eg- $R = \begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$ ⇒ Column 2 of A is linearly dependent on column 1

☆ → if a column is pivot column then it in Linearly independent of all columns before it

Block Matrix Factorisation Results

1) $M = \begin{bmatrix}A&B\\B&A\end{bmatrix}$ $P = \begin{bmatrix}I&I\\I&-I\end{bmatrix}$ $P^{-1}MP = \begin{bmatrix}A+B&0\\0&A-B\end{bmatrix}$

$P^{-1} = \frac{1}{2}\begin{bmatrix}I&I\\I&-I\end{bmatrix}$

$\boxed{\begin{array}{l}\det M = \det(A+B)\det(A-B)\\ \text{if } AB=BA \Rightarrow \det M = \det(A^2-B^2)\end{array}}$

2) $M = \begin{bmatrix}A&B\\-B&A\end{bmatrix}$ $P = \begin{bmatrix}I&iI\\I&-iI\end{bmatrix}$ $P^{-1}MP = \begin{bmatrix}A+iB&0\\0&A-iB\end{bmatrix}$

$\boxed{\det M = \det(A+iB)\det(A-iB)}$

3) $M = \begin{bmatrix}aI&bI\\bI&aI\end{bmatrix}$ $\det M = \det(A+B)\cdot\det(A-B)$ format $= (a+b)^n(a-b)^n$

4) general $\begin{bmatrix}A&B\\C&D\end{bmatrix} \longrightarrow \begin{bmatrix}A&B\\0&D-CA^{-1}B\end{bmatrix}$ $\det = \det(A)\cdot\det(D-CA^{-1}B)$

Trace properties

1. $tr(A) = \sum_{i=1}^{n} a_{ii}$
2. $tr(A+B) = tr(A) + tr(B)$
3. $tr(A^T) = tr(A)$
4. $tr(A^*) = \overline{tr(A)}$
5. $A^T = -A \Rightarrow tr(A) = 0$
6. $tr(I) = n$
7. $tr(P) = rank(P) = \dim C(P) \to$ projected space
8. Nilpotent $A^k = 0 \Rightarrow tr(A) = 0$ (all eigen values 0)

Note : $tr(\log A) = \log[\det(A)]$

9. $tr(AB) = tr(BA)$
10. $tr(ABC) = tr(BCA) = tr(CAB)$ {Cyclic order preserves trace
    $tr(ABC) \neq tr(ACB)$
11. Commutator $[A,B] = AB - BA$
    $tr([A,B]) = tr(AB - BA) = 0$
12. Similar $B = P^{-1}AP$
    $\Rightarrow tr(B) = tr(A)$
13. $tr(A^2) = \sum \lambda_i^2$
14. $tr(A^TA) = \sum_{i,j} a_{ij}^2 \to$ Real case ☆
    ↳ of A

☆

↳ $tr(A^TA) = \sum_{ij} a_{ij}^2 = tr(AA^T) = \sum \sigma_i^2$

Count with algebric multiplicity → sum of singular values

15. $tr(A^{-1}) = \sum_i 1/\lambda_i$
16. $|tr\, A| \le rank(A)\, \|A\|$

☆ 17. Idempotent $A^2 = A \Rightarrow tr(A) = Rank(A)$
- ↳ All eigen value 1 — r multiplicity
  0 — n-r multiplicity
- ↳ Always diagonalisable ☆

18. Trace of Block Matrix

$$A = \begin{bmatrix} B & * \\ * & C \end{bmatrix}$$

$tr(A) = tr(B) + tr(C)$

19. $tr(x^TAx) = x^TAx$

$tr(A^TA) \ge 0$

$$\boxed{\text{20) Frobenius Inner product: } tr(A^TB) = \sum_{ij} a_{ij} b_{ij}}$$

21) Frobenius Norm = Square root of frobenius Inner product

$$= \sqrt{\sum_{ij} a_{ij}^2} = \sqrt{Tr(A^TA)} = \sqrt{Tr(AA^T)}$$

$$= \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots \sigma_r^2} =$$

22) If for 2x2 $-trace(A) = 0 \Rightarrow A^2 = -|A|\, I$ —— from CH Theorem

$A^2 - trace(A) + |A| = 0$

$A^2 = -|A|\, I$

1. Definition (precise)

A **principal minor** is obtained by:

- choosing an index set $I \subset \{1, 2, \dots, n\}$
- keeping rows in I
- keeping columns in I & then taking determinant

The indices **do not have to be consecutive**.

Any diagonal entry < 0	✗ Not PSD, ✗ Not PD
Any diagonal entry = 0	Possibly PSD, ✗ Not PD
Any principal minor < 0	✗ Not PSD, ✗ Not PD
All principal minors ≥ 0	✓ PSD
All leading principal minors > 0	✓ PD
All leading principal minors ≥ 0	✗ No conclusion
All eigenvalues ≥ 0	✓ PSD
All eigenvalues > 0	✓ PD

2. Leading principal minors (exact definition)

For an $n \times n$ matrix A, the **leading principal minors** are:

$$\det(A_1),\ \det(A_2),\ \dots,\ \det(A_n)$$

where A_k is the **top-left** $k \times k$ **submatrix** of A:

$$A_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$$

The characteristic equation of a 3x3 matrix A is given by the formula:
$\lambda^3 - (\text{Trace } \mathbf{A})\lambda^2 + (\text{Sum of principal minors of order 2})\lambda - (\det \mathbf{A}) = \mathbf{0}$.

What are the possible values for which A is PSD

$$\begin{bmatrix} \alpha & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \beta \end{bmatrix}$$

Answer) $S = \{(\alpha, \beta) \mid A \text{ is PSD}\}$

$S = \phi$

$$PM_{1,2} = \underbrace{\begin{vmatrix} \alpha & 1 \\ 1 & 0 \end{vmatrix}}_{\text{submatrix } 1,2} = \underline{\underline{-1}} \quad \text{not PSD}$$

$$\boxed{\begin{array}{l} \text{All leading PM} > 0 \rightarrow \text{PD} \\ \text{All Principle m} \geq 0 \rightarrow \text{PSD} \end{array}}$$

$\lambda^2 - tr(A)\lambda + |A|$

$$\lambda^3 - \lambda^2 tr(A) + \lambda \sum_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} P_{ij} + |A| = 0$$

$P_{ij} = |$ Matrix obtained by choosing i,j rows & columns $|$

Q) Find Sum of elements of $\bar{A}^{1}$ if $A = BCD$

Solution

$$y = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = D^{-1}C^{-1}B^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow BCD\,[y] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{solve}$$

Answer $= \underline{\underline{2}}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 2 & & & \\ & 1 & & \\ & & -2 & \\ & & & 3 \end{bmatrix}}_{D}$$

$x_1 = 2.5$
$x_2 = -1$
$x_4 = 0$
$x_3 = 0.5$

$$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & \cancel{1}2 & 0 & 3 \\ 0 & -2 & -2 & 0 \\ \cancel{2}0 & \cancel{4}0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2\,\cancel{1} \\ 1 \\ \cancel{1}0 \end{bmatrix}$$

Simultaneous Diagonalisation & AB=BA commutation

① If $AB=BA$ & A is diagonalisable $\Rightarrow$ eigen space of A corresponding to any eigen vector λ of A $E_\lambda(A)$ is B-invariant

proof

$AB = BA$

take any eigen vector x of $E_\lambda(A)$

$$ABx = BAx$$
$$A(Bx) = B\lambda x$$
$$A(\underline{Bx}) = \lambda(\underline{Bx})$$

Bx is a vector, it is also an eigen vector of A.

$\Rightarrow$ Bx is in same eigen space.

Sidequest — Invariant definition

$\rightarrow$ A subspace V is M-invariant if

$\forall_{x\in V}\ Mx \in V$

$\forall_x (x\in V \rightarrow Mx \in V)$

$\rightarrow$ But doesn't it imply all eigen vectors are same $\Rightarrow$ No

Matrix B may not be diagonalisable and in the worst case only have one distinct eigen vector space.

$\Rightarrow$ if $AB=BA$ & A diagonalisable $\Rightarrow$ share atleast one eigen vector

$\rightarrow$ if both A & B diagonalisable $\Rightarrow$ Now A & B both have distinct eigen vectors

$\Rightarrow$ All eigen vectors should be same for B invariant eigen space of $E_\lambda(A)$

$\boxed{\text{All eigen vectors same}}$

$\Rightarrow$ A & B are called simultaneusly diagonalisable

$$\left.\begin{aligned} A &= P\Lambda_A P^{-1} \\ B &= P\Lambda_B P^{-1} \end{aligned}\right\}$$

same P
different diagonal matrix Λ_A, Λ_B